UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH2201**

ASSESSMENT PATTERN : **MATH2201A**

MODULE NAME : **Algebra 3: Further Linear Algebra**

DATE : **13-May-10**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH2201A-001-EXAM-186

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Calculate $hcf(68, 12)$ and find integers h and k such that

$$hcf(68, 12) = 68h + 12k$$

   (b) Do the following equations $68x + 12y = 4$ and $68x + 12y = 6$ have integer solutions ? If yes, find them all.

   (c) State the Chinese remainder theorem.
   Find the unique $[z]$ in $\mathbb{Z}/105$ such that $z \equiv 3 \bmod 21$ and $z \equiv 7 \bmod 5$.

2. (a) Let k be a field. Say what is meant by Euclidian division and Bézout's identity for two polynomials f and g in $k[x]$.

   (b) Say what is meant by an irreducible polynomial and state the unique factorisation theorem for polynomials.

   (c) Factorise $f(x) = x^3 - 1$ into irreducibles in $\mathbb{C}[x]$, $\mathbb{R}[x]$, $\mathbb{F}_3[x]$ and $\mathbb{F}_2[x]$.
   In each case you must justify why your factors are irreducible.

   (d) Let V be a vector space over k and let $T: V \longrightarrow V$ be a linear map. Say what is meant by the minimal polynomial m_T of T.
   Show that if $f \in k[x]$ is a non-zero polynomial such that $f(T) = 0$, then m_T divides f.

3. Let V be a vector space over a field k, $T\colon V \longrightarrow V$ a linear map and $\lambda_1, \dots, \lambda_r$ its eigenvalues in k.

(a) Say what is meant by generalised eigenspaces $V_{b_i}(\lambda_i)$.

(b) Show that

$$V_{b_i}(\lambda_i) \subseteq V_{b_i+1}(\lambda_i)$$

and that

$$T(V_{b_i}(\lambda_i)) \subseteq V_{b_i}(\lambda_i)$$

(c) Say what is meant by T being diagonalisable and state the criterion for diagonalisability involving the minimal polynomial.

(d) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Find the minimal polynomial of A.
Is A diagonalisable over $\mathbb{C}$, $\mathbb{R}$, $\mathbb{F}_2$ and $\mathbb{F}_3$?

(e) Let V be a vector space over k and T_1 and T_2 be two linear maps with the same minimal polynomial $m_{T_1}(x) = m_{T_2}(x) = x - \lambda$.
Show that *for any* basis B of V, one has

$$[T_1]_B = [T_2]_B$$

4. In this question k is any field.

(a)

$$A = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix}$$

Find the minimal polynomial, generalised eigenspaces, Jordan basis and Jordan normal form.

(b) In each of the following cases, find the Jordan normal form :

(i) $ch_T = (x-5)^3$ and $m_T = x - 5$

(ii) $ch_T = (x-5)^3$ and $\dim V_1(5) = 2$

(iii) $ch_T = (x-5)^3$ and $m_T = (x-5)^3$

(iv) $ch_T = (x-5)^2(x-4)^2$ and $m_T = (x-5)^2(x-4)$ and $\dim V_1(4) = 2$.

5. Let V be a vector space over $\mathbb{R}$. Let $f: V \times V \longrightarrow \mathbb{R}$ be a bilinear symmetric form and $q(v) = f(v, v)$ the corresponding quadratic form.

   (a) Give, without proof, an expression for $f(v, w)$ in terms of $q(v)$, $q(w)$, $q(v+w)$. Hence deduce that if f is a *non-zero* form, then there exists a vector v such that $q(v) \neq 0$.

   (b) Say what is meant by an orthogonal basis with respect to f.
   Show that there exists an orthogonal basis (you can use **without** proof the fact that if v is such that $q(v) \neq 0$, then $V = Span\{v\} \oplus \{v\}^{\perp}$).

   (c) Say what is meant by the canonical form of q, signature and rank.
   Use part (b) to show that every quadratic form over $\mathbb{R}$ has a canonical form (you are not asked to prove the uniqueness).

   (d) Determine canonical form, rank and signature of the following real form :

$$q(x, y, z) = x^2 + 2xy + 4y^2$$

6. (a) Let $(V, (,))$ be an inner product space over $\mathbb{C}$. Let $T: V \longrightarrow V$ be a linear map. Say what is meant by the adjoint T^* of T.
   Show that if T^* and T' are two adjoints, then $T^* = T'$.

   (b) Let $T: V \longrightarrow V$ be a linear map such that $T^* = T$. Show that the eigenvalues of T are real.
   Show that eigenvectors corresponding to two different eigenvalues are orthogonal.

   (d) Suppose that T is such that $T^*T = 0$. Show that $T = 0$.

END OF PAPER